

Waveshaping

The term waveshaping refers to a conceptually very simple signal processing technique: we take the instantaneous value of the input signal x and apply an arbitrary function $y = f(x)$ to obtain the output value y for the same time instant. The function which takes the input x and produces the output y is called transfer function. In contrast to a filter, a waveshaper does not have memory because the instantaneous output depends solely on the instantaneous input and not on past inputs or output values. In this respect the waveshaper seems to be simpler than a filter. However, because the function $f(x)$ can be basically anything (within reason), waveshaping is an intrinsically nonlinear process. A key feature of nonlinear processes is, that they can create new frequency components which were not present in the input signal. When the input to the waveshaper is a sinusoid with frequency f , its output will be in general a sum of sinusoids with frequencies at integer multiples of f . These additional frequencies are called harmonics. How they come about can easily be understood by considering a simple hard clipper, which just clips the input signal off at some threshold amplitude. Look at figure 1 (a): When we send a sine-wave at

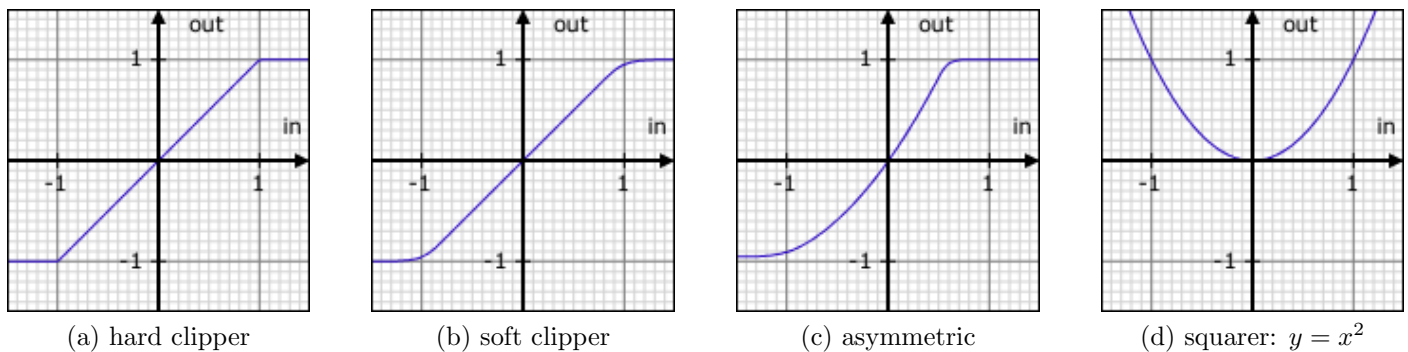


Figure 1: Some examples for waveshaping transfer functions

some frequency through the hard clipper and turn up the amplitude of the input sine-wave, the output will more and more resemble a square wave. Now it is well known from Fourier analysis, that the square wave contains frequencies at all odd integer multiples of the fundamental frequency. So we obviously have created new frequencies which were not present in the input sinusoid. The clipper makes another feature of waveshaping apparent: the spectrum of the output signal depends on the amplitude of the input signal. Driving the hard clipper with a very loud sinusoid results in a rich output spectrum (almost that of a square wave). On the other hand, driving it with a sinusoid of unit amplitude or below will just result in the same sinusoid at the output, which has no harmonics at all (except the fundamental, of course).

Polynomials

A special family of transfer functions which are interesting for waveshaping are polynomials. A polynomial of order n is defined as:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = \sum_{k=0}^n a_k x^k \quad (1)$$

These polynomial functions allow for a somewhat more predictable result than arbitrary functions. Specifically, if we use a polynomial of order n , we will create only harmonics up to order n . This is a convenient

property of polynomials in the context of waveshaping. On the other hand, an inconvenient property is, that polynomials diverge to plus or minus infinity when the absolute value of the input argument becomes arbitrarily large - so we should restrict the input amplitude in practice. To see how polynomial waveshapers create harmonics, we will consider the simplest nonlinear example of the quadratic transfer function $y = x^2$. The waveshapers output signal is therefore: $y(t) = f(x(t)) = (x(t))^2$. Plugging in a sinusoidal input with radian frequency ω and amplitude A : $x(t) = A \cdot \sin(\omega t)$ as input signal, we obtain:

$$\begin{aligned} y(t) &= (A \cdot \sin(\omega t))^2 \\ &= A^2 \cdot \sin^2(\omega t) \end{aligned} \quad (2)$$

Using the trigonometric identity $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$, we obtain:

$$\begin{aligned} y(t) &= A^2 \cdot \frac{1}{2}(1 - \cos(2\omega t)) \\ &= \frac{A^2}{2} - \frac{A^2}{2} \cos(2\omega t) \end{aligned} \quad (3)$$

which we recognize as a constant (i.e. a DC component) and a sinusoid at twice the frequency (albeit with a different phase). There are more general trigonometric identities for higher powers of sinusoidal functions which could be used for predicting the output signal of higher power waveshapers. We wont go through some more boring algebra here to demonstrate this - the general outcome will be a combination of sinusoids of frequencies at integer multiples of the input frequency, up to the highest power in the polynomial.

Chebyshev Polynomials

Inside the class of polynomial waveshapers, another specialization are those polynomials, which wiggle up and down between -1 and +1 as the input signal goes through the domain $-1 \dots +1$. These special polynomials are known as Chebyshev polynomials. The Chebyshev polynomial of order n (denoted as $T_n(x)$) goes through the range $-1 \dots +1$ exactly n times as the input x goes through the domain $-1 \dots +1$. Figure 2 illustrates this. Often when discussing Chebyshev polynomials, one considers only the domain $-1 \dots +1$, outside this domain, they diverge to plus or minus infinity. In the context of waveshaping, Chebyshev polynomials have a further convenient property as opposed to the general polynomials, namely: when using a Chebyshev polynomial of order n as transfer function and using a unit amplitude sinusoid as input we will see *only* the n -th harmonic in the output signal. For the general polynomial of order n we would have seen a combination of harmonics *up to* order n . When using linear combinations of Chebyshev polynomials, we can conveniently compose any desired combination of harmonics. Note, however, that the statement above does hold only for unit amplitude sinusoids - when the amplitude is not unity, we will see a combination of harmonics at the output of Chebyshev polynomials as well.

Symmetric Transfer Functions / Even and Odd Harmonics

When we look - for example - at the Chebyshev polynomials, we observe certain symmetries. The odd order polynomials possess a point-symmetry about the origin whereas the even order polynomials possess axial symmetry about the y-axis. In the former case, the identity $f(x) = -f(-x)$ holds and this kind of symmetry is called odd symmetry in mathematics. In the latter case, the identity $f(x) = f(-x)$ holds and

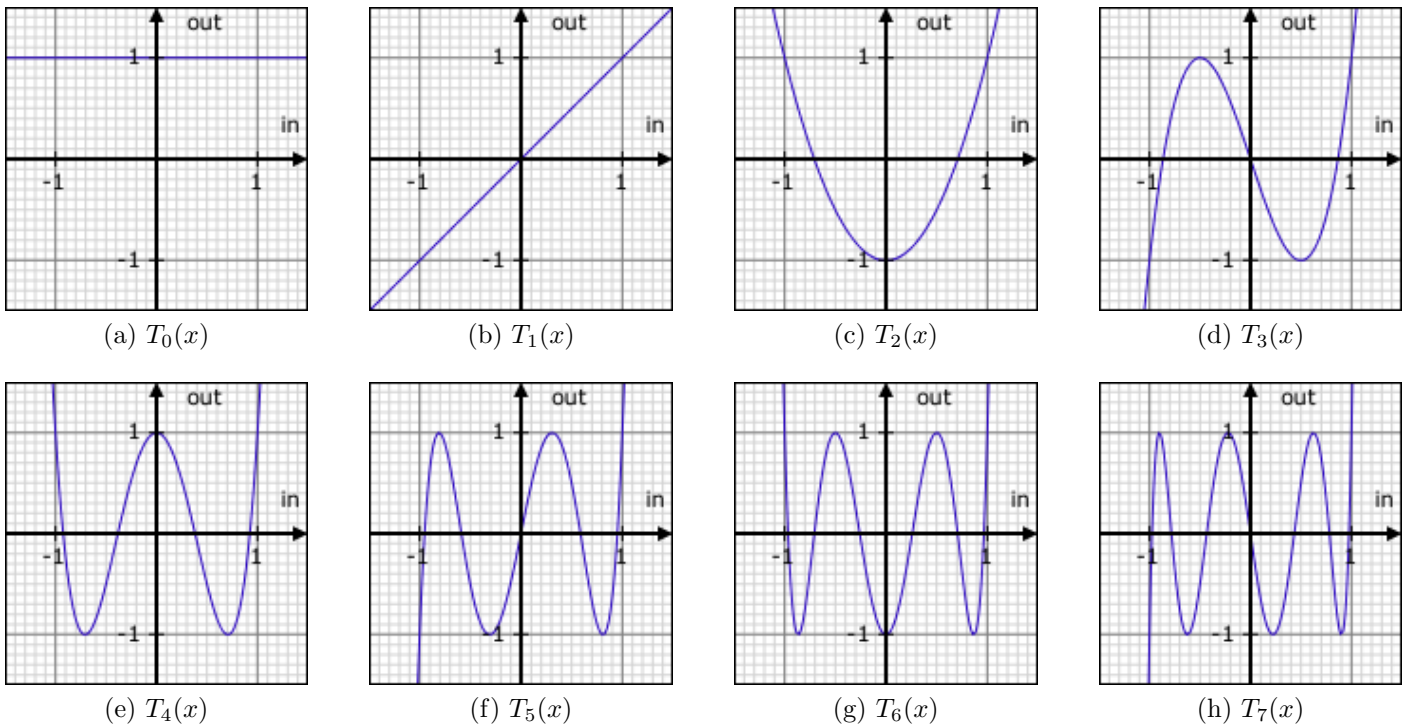


Figure 2: Chebyshev Polynomials of order 0 - 7

this kind of symmetry is called even symmetry. Such symmetries have implications on the harmonics which will be generated by our transfer function, namely: transfer functions with odd symmetry will create only odd order harmonics (frequencies at odd integer multiples of the input sinusoid) whereas transfer functions with even symmetry will create only even order harmonics. This statement holds for any function which possesses the respective symmetry, regardless of whether it is a polynomial or not. The hard clipper, for example has, odd symmetry and will create only odd harmonics therefore. A transfer function that does neither have even nor odd symmetry, will create even as well as odd harmonics in the output signal.

Intermodulation and Powerchords

Up to now, we considered the simple case where the input signal to the waveshaper is a pure sinusoid. Now we turn to the question what happens when the input signal is more complex, i.e. a sum of sinusoids. The simplest case here would be a sum of just two sinusoids. What could be expected is that the output signal contains the harmonics for both of the input sinusoids - which is true. However, this is only half of the story. In addition to these two harmonic series, we will also see sum- and difference-frequencies. To see what this means, let's consider once more the simple quadratic polynomial $y = f(x) = x^2$ as transfer function, but this time we choose as input signal $x(t)$ the sum of two sinusoids with radian frequencies ω_1, ω_2 and amplitudes A_1, A_2 : $x(t) = A_1 \cdot \sin(\omega_1 t) + A_2 \cdot \sin(\omega_2 t)$, so we have:

$$\begin{aligned}
 y(t) &= (A_1 \cdot \sin(\omega_1 t) + A_2 \cdot \sin(\omega_2 t))^2 \\
 &= A_1^2 \cdot \sin^2(\omega_1 t) + A_2^2 \cdot \sin^2(\omega_2 t) + 2A_1 A_2 \sin(\omega_1 t) \cdot \sin(\omega_2 t)
 \end{aligned}
 \tag{4}$$

The first two terms are of the same form as in equation 2, hence, they will create the DC components and components at $2\omega_1$ and $2\omega_2$ as before. The third term, however, represents a new phenomenon. To see the frequency components, which are created by this term, we need a further trigonometric identity: $\sin(\alpha) \cdot \sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$. Using this identity and simplifying, we obtain:

$$y(t) = \frac{A_1^2}{2} - \frac{A_1^2}{2} \cos(2\omega_1 t) + \frac{A_2^2}{2} - \frac{A_2^2}{2} \cos(2\omega_2 t) + A_1 A_2 \cos(\underbrace{(\omega_1 - \omega_2)}_{\omega_d} t) - A_1 A_2 \cos(\underbrace{(\omega_1 + \omega_2)}_{\omega_s} t) \quad (5)$$

where ω_s and ω_d are the sum- and difference-frequencies respectively. As a concrete example, let the frequencies of our two sinusoids are $1000Hz$ and $300Hz$ and the amplitudes both be unity. According to the math, the output signal of the squarer contains DC with unit amplitude, $2kHz$ and $600Hz$ both with amplitude $\frac{1}{2}$ and $1300Hz$ and $700Hz$ with unit amplitude.

For a more complex mix of sinusoids at the input and more complex transfer functions, even more of such cross-terms would appear. When this mix of sinusoids at the input is itself a periodic signal (a sum of sinusoids at integer multiples of some fundamental frequency), then those cross-terms will appear at integer multiples of the fundamental and thus coincide (frequency-wise) with the harmonics which are already there. In this case, we will only modify the relative amplitudes of the harmonics but not create something qualitatively new. But when we pass - for example - a chord through a waveshaper, containing the prime, the third and the fifth (each with its own harmonic series) - we usually see (or hear) an incomprehensible spectral mess at the waveshaper's output, due to the various sum- and difference-tones. If only the fifth is present, the result does still sound rather good. Guitarists are familiar with this phenomenon and that's why they rarely play full chords over a distorted amp - powerchords usually contain only the fifth, anything else would be too weakly related with the basic pitch and result in muddy output signal.