Basic Digital Filters

One Pole Filter

Difference Equation:

$$y[n] = b_0 x[n] - a_1 y[n-1] \tag{1}$$

Transfer Function:

$$H(z) = \frac{b_0}{1 + a_1 z^{-1}} \tag{2}$$

Complex Frequency Response:

$$H(e^{j\omega}) = \frac{b_0}{1 + a_1 e^{-j\omega}} \tag{3}$$

Magnitude Response:

$$|H(e^{j\omega})| = \sqrt{\frac{b_0^2}{1 + a_1^2 + 2a_1 \cos(\omega)}} \tag{4}$$

Phase Response:

$$\angle H(e^{j\omega}) = \begin{cases} atan2(a_1 \sin(\omega), 1 + a_1 \cos(\omega)) & \text{for } b > 0 \\ atan2(a_1 \sin(\omega), 1 + a_1 \cos(\omega)) + \pi & \text{for } b < 0 \end{cases} \tag{5}$$

Real Part:

$$\Re\{H(e^{j\omega})\} = \frac{b_0 + a_1 b_0 \cos(\omega)}{1 + a_1^2 + 2a_1 \cos(\omega)} \tag{6}$$

Imaginary Part:

$$\Im\{H(e^{j\omega})\} = \frac{a_1 b_0 \sin(\omega)}{1 + a_1^2 + 2a_1 \cos(\omega)} \tag{7}$$

First Order Filter

Difference Equation:

$$y[n] = b_0 x[n] + b_1 x[n-1] - a_1 y[n-1] \tag{8}$$

Transfer Function:

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \tag{9}$$

Complex Frequency Response:

$$H(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}} \tag{10}$$

Magnitude Response:

$$|H(e^{j\omega})| = \sqrt{\frac{b_0^2 + b_1^2 + 2b_0 b_1 \cos(\omega)}{1 + a_1^2 + 2a_1 \cos(\omega)}} \tag{11}$$

Phase Response:

$$\angle H(e^{j\omega}) = \arctan\left(-\frac{(b_1 - a_1 b_0)\sin(\omega)}{b_0 + a_1 b_1 + (b_1 + a_1 b_0)\cos(\omega)}\right) \tag{12}$$

Real Part:

$$\Re\{H(e^{j\omega})\} = \frac{b_0 + a_1 b_1 + (b_1 + a_1 b_0)\cos(\omega)}{1 + a_1^2 + 2a_1 \cos(\omega)} \tag{13}$$

Imaginary Part:

$$\Im\{H(e^{j\omega})\} = -\frac{(b_1 - a_1 b_0)\sin(\omega)}{1 + a_1^2 + 2a_1 \cos(\omega)} \tag{14}$$

Biquad

Difference Equation:

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] - a_1 y[n-1] - a_2 y[n-2] \tag{15}$$

Transfer Function:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{16}$$

Complex Frequency Response:

$$H(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + b_2 e^{-2j\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega}} \tag{17}$$

Magnitude Response:

$$|H(e^{j\omega})| = \sqrt{\frac{b_0^2 + b_1^2 + b_2^2 + 2(b_0 b_1 + b_1 b_2)\cos(\omega) + 2b_0 b_2 \cos(2\omega)}{1 + a_1^2 + a_2^2 + 2(a_1 + a_1 a_2)\cos(\omega) + 2a_2 \cos(2\omega)}} \tag{18}$$

General Direct Form Filter

Difference Equation:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \tag{19}$$

Transfer Function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{20}$$

Let:

$$c_a = 1 + \sum_{k=1}^{N} a_k \cos(k\omega), \quad s_a = \sum_{k=1}^{N} a_k \sin(k\omega), \quad c_b = \sum_{k=0}^{M} b_k \cos(k\omega), \quad s_b = \sum_{k=0}^{M} b_k \sin(k\omega) \tag{21}$$

Complex Frequency Response:

$$H(e^{j\omega}) = \frac{c_b - j s_b}{c_a - j s_a} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\omega}} \tag{22}$$

Magnitude Response:

$$|H(e^{j\omega})| = \sqrt{\frac{c_b^2 + s_b^2}{c_a^2 + s_a^2}} \tag{23}$$

Phase Response:

$$\angle H(e^{j\omega}) = \operatorname{atan2}(s_b, c_b) - \operatorname{atan2}(s_a, c_a) \tag{24}$$